Derivation for Miner's Rule – Its uses, Its Assumptions, and A few examples

Understanding Cyclic Size and Thermodynamic Work

In order to derive Miner's rule, one must first understand the concepts of cyclic size and thermodynamic work. Cyclic work can be illustrated graphically as shown in Figure 1 (Ref. 1) where the generalized displacement X and force Y have been employed. The work "areas" in Fig. 1 describe the work done per cycle on the system. For example, for the paper clip, the coordinates are stress and strain (the forces constitute the strain).

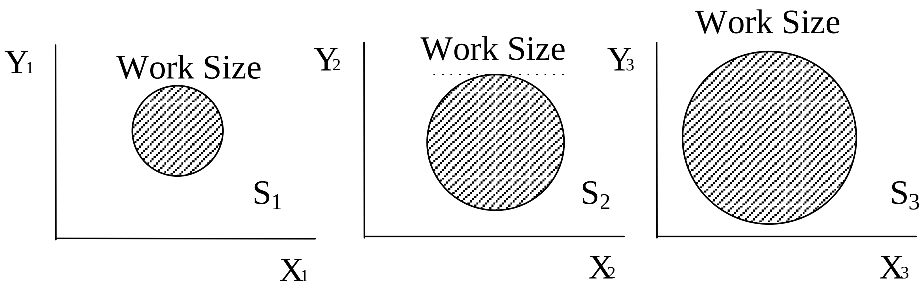

Figure 1 Cyclic work plane of three different sizes.

Cyclic work is repetitive and can be summed as a measure of the total cyclic work performed. In this chapter, we define cumulative damage by the ratio of the sum of the thermodynamic work performed per cycle to the total amount of cyclic work required to cause failure as

$$Damage = \frac{\text{Thermodynamic Work for n Cycles}}{\text{Themodynamic Work to Failure}}$$

A Derivation of Miner's Rule[1]

We start with the thermodynamic work W that is a function of the cyclic size S and the number of cycles n such that

$$W_n = W(S, n)$$

If we assume that the work for n cycles of the same size obeys the relationship

$$W_n = n\,W(S)$$

failure will occur suddenly after N cycles. Substituting this into the equation for damage, and summing over cycles of possibly different size yields

$$Damage = \frac{n_1 W(S_1) + n_2 W(S_2) + ...}{W_{Failure}} = \frac{n_1 W(S_1)}{W_{Failure}} + \frac{n_2 W(S_2)}{W_{Failure}} + ...$$

If N_j denotes the number of cycles of size S_j to failure, then the identity

$$W_{Failure} = N_1 W_1(S_1) = N_2 W_2(S_2) = N_3 W_3(S_3) = ...$$

allows us to obtain the classical fatigue equation due to Miner for cumulative damage at each ith stress level that reads

$$Damage = \frac{n_1}{N_1}\frac{W(S_1)}{W(S_1)} + \frac{n_2}{N_2}\frac{W(S_2)}{W(S_2)} + \frac{n_3}{N_3}\frac{W(S_3)}{W(S_3)} + ... = \frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + ... = \sum_i \frac{n_i}{N_i}$$

This derivation which is not in Miner's original paper (and can only be found in Ref. 1) helps in understanding the use of $W_n = n\, W(S)$ in some high-stress applications.

EXAMPLE 14.2: Miner's Rule

A pressure vessel is made of aluminum alloy and operates in two states at 20 cycles per minute. It operates in the first state 60% of the time at a stress change of 1500 psi; the cycles to failure in this state are 1,000,000 cycles. In state 2 (40% of the time), a stress change of 2400 psi is exerted on the vessel, and the cycles to failure in state 2 is 215,000. The problem is to find the expected life in hours of this hydraulic pressure unit.

SOLUTION: The information is $n_1+n_2=20$ cycles per minute, or effectively
$n_1= 0.6 \times 20= 12$ cycles per minute and $n_2= 0.4 \times 20 = 8$ cycles per minute

From Miner's rule, we have

$$Damage = \sum_i \frac{n_i}{N_i} = \frac{n_1}{N_1} + \frac{n_2}{N_2}$$

which is the portion of total life consumed per minute of operation. The damage per unit time is

$$Damage = \frac{12}{1{,}000{,}000} + \frac{8}{215{,}000} = 0.49 x 10^{-4} \text{ per minute} = 0.00294 \text{ per hour}$$

Since failure occurs when damage = 1, then
Damage = 1 = (at N hours for failure) × (0.00294 amount of damage per hour)
or solving for N hours is
N hours for failure=1/0.00294=340 hours.
This example may be found in Miner's original paper in Reference 3.

EXAMPLE 14.3

Aluminum alloy has the following fatigue characteristics:
Stress 1, N = 45 cycles,
Stress 2, N = 310 cycles, and
Stress 3, N = 12,400 cycles.
How many times can the following sequence be repeated?
n_1 = 5 cycles at stress 1,
n_2 = 60 cycles at stress 2, and
n_3 = 495 cycles at stress 3.

SOLUTION: The fractions of life exhausted in each block are

$$\frac{n_1}{N_1} = \frac{5}{45} = 0.111, \quad \frac{n_2}{N_2} = 0.194, \quad \frac{n_3}{N_3} = 0.04$$

The fraction of life exhausted in a complete sequence is approximately 0.345. The life is entirely exhausted when damage D = 1, and at this point the sequence is repeated X times; X(0.345) = 1. Solving for X gives X = 2.9 times. For example, after the sequence is repeated twice, the fraction of life exhausted is 0.69. Then n_1 and n_2 would bring it up to about 0.995, which is close enough for failure to occur.

Assumption in Miner's Rule

Miner's rule applies if cyclic work can be approximated by the number of cycles times an average amount of work per cycle of a certain size [i.e. $W_n = nW(S)$]. However, if the work is some nonlinear function of the number of cycles, then Miner's rule is invalid. If the paper clip bends past its elastic limit in some of the stress cycles, then W_n is nonlinear in the material. This corresponds to over-stressing the material. If the material is severely over-stressed during each cycle, then the approximation $W_n = nW(S)$ used in the derivation of Miner's rule is not reasonable. In this case, the amount of work to bend the paper clip at the same stress level will be largely reduced from cycle to cycle and W_n is nonlinear in n. Overall, the assumptions follow:

- Minor's Rule Assumes:
  $\text{Work} = W(S_i)\, n$

- Sometimes Require:
  $\text{Work} = W(S_i, n)$

Figure 2 Assumption of Miners Rule.

In the latter case, the work could be incrementally summed with the actual stress applied to the material. Damage is then expressed in terms of the generalized coordinates in as

$$Damage = \frac{\sum_n \oint Y_n \, dX_n}{W_{Failure}}$$

Reference:
1. Design for Reliability, Published By CRC Press, Edited by A. Feinberg, D. Crow (Chapter 14)